Spherical wave functions and dyadic Green's functions for homogeneous elastic anisotropic media

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(Received 19 October 1992)

A spherical-wave-function theory of bounded homogeneous elastic anisotropic media is developed. The anisotropic elastodynamic wave equations are solved exactly by using the method of plane-wave angular spectrum expansions. The series, integral representations, and addition theorems of the spherical wave functions of the first, second, third, and fourth kind for homogeneous elastic anisotropic media (HEAM) are presented. Weyl's method for the scalar Green's functions in isotropic media is generalized to the study of the dyadic Green's functions in HEAM. The series representations of Green's functions are of the form of separated variables. These representations are well suited to imposing a boundary condition when dealing with waves in spherically layered HEAM.

PACS number(s): 43.20.+g, 03.40.Kf

I. INTRODUCTION

The spherical wave functions (SWFs) for homogeneous elastic isotropic media (HEIM) are well known [1]. The eigenfunctions play a very important role in solving the elastodynamic wave problems in both isotropic and anisotropic media. In order to apply the efficient recursive algorithm developed by Chew and co-workers to the elastic wave scattering by many scatterers and multilayered scatterers of elastic anisotropic media [2] and apply the multiple scattering theory to the discrete random media [3], vector wave functions of all kinds and their addition theorems are required. However, to the best of our knowledge, so far the corresponding treatment of eigenfunctions for homogeneous elastic anisotropic media (HEAM) has not been given.

The point-source radiation in HEAM is an important research subject in linear and nonlinear acoustics [4]. The far-region fields can be calculated by the saddle-point method [5], which has been used in the analysis of many nonlinear phenomena [4]. We need the general representations of the dyadic Green's functions to establish the boundary integral equations [5]. We also need dyadic Green's functions in the form of separable variables to derive the T-matrix formulation from Huygen's principle and the extinction theorem of HEAM [6]. However, the general methods for the common representations of Green's dyadic for HEAM have not been published yet except for those in the rectangular coordinate system and in the Fourier transform domain [5].

Recently, there has been an increasing research interest in the theory of bounded HEAM [6–8]. The elastic wave theory of unbounded HEAM is well known [4–9]. However, the bounded cases can only be numerically treated by some methods, such as finite element, finite difference, moment, etc. [3].

This paper is the outgrowth of the corresponding progress in electromagnetics [10–19]. In this paper, angular spectrum representations similar to those proved in many papers [10–16] are derived and used as the starting point of our theory. The use of the expansion of a dyadic plane wave [20,21] considerably simplifies the analysis. An elastic wave theory of bounded homogeneous elastic anisotropic media is developed in this paper. The series representations of the first, second, third, and fourth kind of spherical wave functions for homogeneous elastic anisotropic media are obtained. Each term of the series is a product of a spherical harmonic function and a two-dimensional finite-range integral containing a spherical Bessel function. The addition theorem of SWFs for HEAM can be obtained directly from that of SWFs for HEIM. Weyl's method of deriving the scalar Green's function in isotropic media is extended to the Green's functions in elastic anisotropic media. The dyadic Green's functions are of separation-of-variables form.

There are six sections in this paper. Section I summarizes the anisotropic electrodynamic wave equations and plane-wave solutions. In Sec. II we solve the anisotropic elastodynamic wave equations in the spherical coordinate system and give the series representations and the addition theorems of spherical wave functions of the first, second, third, and fourth kind for HEAM. Furthermore, we discuss the integral representations of wave functions in Sec. IV. Section V turns to the evaluation of Green's dyadic for HEAM in a spherical coordinate system. Section VI concludes the work with a discussion of related problems.

II. WAVE EQUATION AND PLANE-WAVE SOLUTION

We shall use the same units and notation as Auld's except for the time dependence. In the following analysis in $\exp(-i\omega t)$ time dependence is assumed and is suppressed throughout. In addition, summation over repeated subscripts is assumed. Although some of these derivations may be referred elsewhere [4,9], we shall summarize them briefly for the sake of completeness.

Assume an anisotropic elastic medium characterized by constitutive relations in a rectangular coordinate system [9],

$$\mathbf{T}=\mathbf{C}:\mathbf{S}\ , \tag{1}$$

1063-651X/93/47(6)/4439(8)/$06.00

$$\mathbf{p}=\rho\mathbf{V} \ , \tag{2}$$

where T, S, $\mathbf{p}$, and $\mathbf{V}$ are the stress field tensor, the strain field tensor, the particle momentum density, and particle velocity, respectively. C and ρ are the stiffness matrix and the density of the matter, respectively. The colon in (1) denotes the double scalar (or double dot) product of a fourth-rank tensor and a second-rank tensor or a third-rank tensor and a vector. All the elements of the matrices are constants in a rectangular coordinate system.

The anisotropic elastodynamic wave equations are given as [9]

$$\nabla\cdot\mathrm{C}:\nabla_s\mathbf{V}-\rho\omega^2\mathbf{V}=\mathbf{O} \ . \tag{3}$$

Physically, when the medium is homogeneous, plane-wave solitons of the form $e^{j(k\hat{l}\cdot\mathbf{r}-\omega t)}$ are admissible, where $\hat{l}$ is a unit vector in the propagation direction. Mathematically, we can solve the elastodynamic wave equations of HEAM in the plane-wave angular spectrum domain by the Fourier transformation [10–16],

$$\mathbf{V}(\mathbf{r})=\int_{-\infty}^{+\infty}d^3k e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{V}(\mathbf{k}) \ , \tag{4a}$$

$$\mathbf{k}=k_x\hat{\mathbf{x}}+k_y\hat{\mathbf{y}}+k_z\hat{\mathbf{z}} \ . \tag{4b}$$

Substituting (4) into (3), we get the following Christoffel equation [9]:

$$k^2 l_{iK}C_{KL}l_{Lj}V_j=\rho\omega^2 V_i \ , \tag{5}$$

where C_{KL} is called elastic stiffness constant and [9]

$$l_z=\cos\varphi_k\sin\theta_k, l_y=\sin\varphi_k\sin\theta_k, l_z=\cos\theta_k \ , \tag{6a}$$

$$l_{iK}=\begin{bmatrix} l_x & 0 & 0 & 0 & l_z & l_y \\ 0 & l_y & 0 & l_z & 0 & l_x \\ 0 & 0 & l_z & l_y & l_x & 0 \end{bmatrix} \ , \tag{6b}$$

$$l_{Lj}=\tilde{l}_{iK} \ , \tag{6c}$$

where the tilde designates a transposed matrix.

Performing the matrix multiplications in (5), we have [9]

$$k^2\begin{bmatrix} \alpha & \delta & \epsilon \\ \delta & \beta & \zeta \\ \epsilon & \xi & \gamma \end{bmatrix}\begin{bmatrix} V_x(\mathbf{k}) \\ V_y(\mathbf{k}) \\ V_z(\mathbf{k}) \end{bmatrix}=\rho\omega^2\begin{bmatrix} V_x(\mathbf{k}) \\ V_y(\mathbf{k}) \\ V_z(\mathbf{k}) \end{bmatrix} \ , \tag{7}$$

where

$$\begin{aligned}
\alpha&=c_{11}l_x^2+c_{66}l_y^2+c_{55}l_x^2+2c_{56}l_yl_z+2c_{15}l_zl_x+2c_{16}l_xl_y \ , \\
\beta&=c_{66}l_x^2+c_{22}l_y^2+c_{44}l_z^2+2c_{24}l_yl_z+2c_{46}l_zl_x+2c_{26}l_xl_y \ , \\
\gamma&=c_{55}l_x^2+c_{44}l_y^2+c_{33}l_z^2+2c_{34}l_yl_x+2c_{35}l_zl_z+2c_{45}l_xl_y \ , \\
\delta&=c_{16}l_x^2+c_{26}l_y^2+c_{45}l_z^2+(c_{46}+c_{25})l_yl_z+(c_{14}\pm c_{56})l_zl_x+(c_{12}+c_{66})l_xl_y \ , \\
\epsilon&=c_{15}l_x^2+c_{46}l_y^2+c_{35}l_z^2+(c_{45}+c_{36})l_yl_z+(c_{13}+c_{55})l_zl_x+(c_{14}+c_{56})l_xl_y \ , \\
\zeta&=c_{56}l_x^2+c_{24}l_y^2+c_{34}l_z^2+(c_{44}+c_{23})l_yl_z+(c_{36}+c_{45})l_zl_x+(c_{25}+c_{46})l_xl_y \ .
\end{aligned} \tag{8}$$

For a nontrivial solution of $\mathbf{V}(\mathbf{k})$, the characteristic determinant of (7) must be equal to zero. This condition yields the dispersion equation [22],

$$-\lambda^3+(\alpha+\beta+\gamma)\lambda^2-\lambda[\alpha\gamma+\beta\gamma+\alpha\beta-\zeta^2+\epsilon^2+\delta^2)]/A+A=0 \ , \tag{9a}$$

$$A=\alpha\beta\gamma+2\delta\zeta\epsilon-a\zeta^2-\beta\epsilon^2-\gamma\delta^2 \ , \tag{9b}$$

$$\lambda=\omega^2\rho/k^2 \ . \tag{9c}$$

The roots of Eq. (9) are designated in the following as $\lambda=\lambda_n$ ($n=1,2,3,\lambda_1<\lambda_2<\lambda_3$) and $k_n=\omega(\rho\lambda_n)^{1/2}$. The corresponding eigenvectors are given by [22]

$$\mathbf{V}_1(\mathbf{k})=\hat{\mathbf{m}}\times\hat{\mathbf{n}} \ , \tag{10a}$$

$$\mathbf{V}_2(\mathbf{k})=\hat{\mathbf{m}}+\hat{\mathbf{n}} \ , \tag{10b}$$

$$\mathbf{V}_3(\mathbf{k})=\hat{\mathbf{m}}-\hat{\mathbf{n}} \ , \tag{10c}$$

$$\hat{\mathbf{m}}=\gamma_3\hat{\mathbf{z}}+\gamma_2\hat{\mathbf{x}}, \quad \hat{\mathbf{n}}=\gamma_3\hat{\mathbf{z}}-\gamma_2\hat{\mathbf{x}} \ , \tag{10d}$$

$$\gamma_3=\left[\frac{\lambda_3-\lambda_2}{\lambda_3-\lambda_1}\right]^{1/2}, \quad \gamma_2=\left[\frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1}\right]^{1/2} . \tag{10e}$$

If the eigenvectors $\mathbf{V}_n(\mathbf{k})$ are found, the stress matrix of the nth eigenwave can be easily calculated by [9]

$$\mathrm{T}_n=-C_{KL}l_{Lj}V_{jn}k/\omega \ . \tag{11}$$

The tensor T_n is obtained by the abbreviated subscript notation introduced in Sec. 1 F of Auld's book [9].

Returning to Eq. (4), it is evident that the integration over the radial wave number is reduced to a summation of three terms which correspond to the three roots k_n ($n=1,2,3$). Therefore, we obtain the eigenwave angular spectrum expansion of the velocity vector inside a spherical region of HEAM as follows [14–16]:

$$\mathbf{V}(\mathbf{r})=\sum_{n=1}^{3}\int_0^{2\pi}d\varphi_k\int_0^{\pi}d\theta_k k_n\sin\theta_k V_{jn}\hat{\mathbf{e}}_j C_n(\mathbf{k})e^{i\mathbf{k}_n\cdot\mathbf{r}} \ . \tag{12a}$$

Similarly, we have the corresponding expression for the stress tensor

$$\mathrm{T}(\mathbf{r})=\sum_{n=1}^{3}\int_0^{2\pi}d\varphi_k\int_0^{\pi}d\theta_k k_n\sin\theta_k T_{ijn}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j C_n(\mathbf{k})e^{i\mathbf{k}_n\cdot\mathbf{r}} \ , \tag{12b}$$

where $\hat{\mathbf{e}}_j$ is the unit vector in the jth direction in a rectangular coordinate system, i.e., $\hat{\mathbf{e}}_1=\hat{\mathbf{x}}$, $\hat{\mathbf{e}}_2=\hat{\mathbf{y}}$, $\hat{\mathbf{e}}_3=\hat{\mathbf{z}}$; $C_n(\mathbf{k})$ is the undetermined amplitude function; k_n, V_{jn}, and T_{ijn} are the eigenwave number, the components of the velocity vector $\mathbf{V}$, and the components of stress tensor T of the nth eigenwave, respectively. The theoretical analysis and the numerical results in Refs. [10–16] give the proofs of (12).

III. SPHERICAL WAVE FUNCTIONS IN HEAM

In this section, the spherical wave functions for HEAM are presented. From the angular spectrum representations (12) of the fields, we obtain the spherical wave function by the spherical harmonic-function expansion of the undetermined amplitudes $C_n(\mathbf{k})$ in conjunction with the spherical wave-function representations of a plane wave. The formulation includes three canonical cases of scalar, vector, and tensor fields in classical continuum physics.

A classical way of expanding the unknown angular spectrum amplitude is to use the series of orthogonal complete harmonic functions on a spherical surface [20]:

$$C_n(\mathbf{k})=\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{+l'} b_{nl'm'}X_{l'm'}(\theta_k,\varphi_k) , \tag{13}$$

where $X_{lm}(\theta,\varphi)=P_l^m(\cos\theta)e^{im\varphi}$, and $P_l^m(\cos\theta)$ is the associated Legendre polynomials. These notations are assumed throughout this paper. The particular form of (12) suggests the use of the identity [20]

$$e^{i\mathbf{k}_n\cdot\mathbf{r}}=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} A_{lm}j_l(k_nr)X_{lm}^*(\theta_k,\varphi_k)X_{lm}(\theta,\varphi) , \tag{14a}$$

$$A_{lm}=i^l(2l+1)\frac{(l-m)!}{(l+m)!} . \tag{14b}$$

Substituting Eqs. (13) and (14) into Eq. (12b), and letting φ be one of the components of T, we find

$$\varphi(\mathbf{r})=\sum_{n=1}^{3}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{+l'} b_{nl'm'}\varphi_{nl'm'}(\mathbf{r}) , \tag{15a}$$

$$\varphi_{nl'm'}(\mathbf{r})=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\int_0^{\pi}\int_0^{2\pi}k_n\sin\theta_k k_n d\theta_k d\varphi_k\varphi_n(\theta_k,\varphi_k)A_{lm}X_{l'm'}(\theta_k,\varphi_k)X_{lm}^*(\theta_k,\varphi_k)j_l(k_nr)X_{lm}(\theta,\varphi) . \tag{15b}$$

Equation (15) is the definition of scalar spherical wave functions of the first kind for HEAM. Because spherical Bessel functions and spherical Hankel functions satisfy the same equation and the same recursive relations, Eq. (15) is also the solution to the problem after the replacement of the spherical Bessel functions of the first kind by the spherical Bessel functions of other kinds (the second, third, and fourth). Thus we give a general definition on scalar spherical wave functions of HEAM as follows:

$$\varphi_{nl'm'}^{(i)}(\mathbf{r})=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}X_{lm}(\theta,\varphi)\left[\int_0^{\pi}\int_0^{2\pi}k_n\sin\theta_k d\theta_k d\varphi_k\varphi_n(\theta_k,\varphi_k)X_{l'm'}(\theta_k,\varphi_k)A_{lm}X_{lm}^*(\theta_k,\varphi_k)z_l^{(i)}(k_nr)\right],\quad i=1,2,3,4 \tag{16}$$

where

$$z_l^{(i)}(k_n)r)=\begin{cases} j_l(k_nr), & i=1\\ y_l(k_nr), & i=2\\ h_l^{(1)}(k_nr), & i=3\\ h_l^{(2)}(k_nr), & i=4\end{cases}$$

is the spherical Bessel function of the ith kind and $\varphi_n(\theta_k,\varphi_k)$ is the known function of the medium parameters as given in Eq. (11).

In order to derive the vector spherical wave functions for HEAM, we use the dyadic plane-wave representations [20,21],

$$\mathsf{I}e^{i\mathbf{k}_n\cdot\mathbf{r}}=\sum_{l,m}A_{lm}\left[-i\mathbf{P}_{lm}(\theta_k,\varphi_k)\mathbf{L}_{lm}^{(1)}(k_n,\mathbf{r})+\frac{1}{\sqrt{l(l+1)}}[\mathbf{C}_{lm}(\theta_k,\varphi_k)\mathbf{M}_{lm}^{(1)}(k_n,\mathbf{r})-i\mathbf{B}_{lm}(\theta_k,\varphi_k)\mathbf{N}_{lm}^{(1)}(k_n,\mathbf{r})]\right] , \tag{17a}$$

$$\begin{aligned}V_{jn}\hat{\mathbf{e}}_je^{i\mathbf{k}_n\cdot\mathbf{r}}&=V_{jn}\hat{\mathbf{e}}_j\cdot\mathsf{I}e^{i\mathbf{k}_n\cdot\mathbf{r}}\\&=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}A_{lm}\Bigg[-iV_{jn}\hat{\mathbf{e}}_j\cdot\mathbf{P}_{lm}(\theta_k,\varphi_k)\mathbf{L}_{lm}^{(1)}(k_n,\mathbf{r})\\&\qquad+\frac{1}{\sqrt{l(l+1)}}[V_{jn}\hat{\mathbf{e}}_j\cdot\mathbf{C}_{lm}(\theta_k,\varphi_k)\mathbf{M}_{lm}^{(1)}(k_n,\mathbf{r})-iV_{jn}\hat{\mathbf{e}}_j\cdot\mathbf{B}_{lm}(\theta_k,\varphi_k)\mathbf{N}_{lm}^{(1)}(k_n,\mathbf{r})]\Bigg] ,\end{aligned} \tag{17b}$$

where A_{lm} is given in Eq. (14b), **P**, **B**, and **C** (the vector functions on a spherical surface) as well as the vector spherical wave functions $\mathbf{L}^{(i)}$, $\mathbf{M}^{(i)}$, and $\mathbf{N}^{(j)}$ $(i=1,2,3,4)$ are defined in the Appendix.

Substituting (17b) and (13) into (12a), we obtain the vector spherical wave functions of the first kind of HEAM as follows:

$$\mathbf{V}(\mathbf{r})=\sum_{n-1}^{3}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{+l'} b_{nl'm'}\mathbf{V}^{(1)}_{nl'm'}(\mathbf{r}) \ , \tag{18a}$$

$$\mathbf{V}^{(1)}_{nl'm'}(\mathbf{r})=\int_0^{\pi}\int_0^{2\pi}X_{l'm'}(\theta_k,\varphi_k)\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}A_{lm}\Bigg[-iV_{jn}\hat{\mathbf{e}}_j\cdot\mathbf{P}_{lm}(\theta_k,\varphi_k)\mathbf{L}^{(1)}_{lm}(k_n,\mathbf{r}) + \frac{1}{\sqrt{l(l+1)}}[V_{jn}\hat{\mathbf{e}}_j\cdot\mathbf{C}_{lm}(\theta_k,\varphi_k)\mathbf{M}^{(1)}_{lm}(k_n,\mathbf{r}) - iV_{jn}\hat{\mathbf{e}}_j\cdot\mathbf{B}_{lm}(\theta_k,\varphi_k)\mathbf{N}^{(1)}_{lm}(k_n,\mathbf{r})]\Bigg] \ . \tag{18b}$$

As vector spherical wave functions $\mathbf{L}^{(i)}$, $\mathbf{M}^{(i)}$, $\mathbf{N}^{(i)}$ $(i=1,2,3,4)$ satisfy the same vector wave equations and the same recursive relations as $\mathbf{L}^{(1)}$, $\mathbf{M}^{(1)}$, $\mathbf{N}^{(1)}$, Eqs. (18) also satisfy the vector wave equation (3) after the replacement of $\mathbf{L}^{(1)}$, $\mathbf{M}^{(1)}$, and $\mathbf{N}^{(1)}$ by $\mathbf{L}^{(i)}$, $\mathbf{M}^{(i)}$, and $\mathbf{N}^{(i)}$ $(i=2,3,4)$, respectively. Finally, we achieve our goal

$$\mathbf{V}^{(i)}_{nl'm'}(\mathbf{r})=\int_0^{\pi}\int_0^{2\pi}X_{l'm'}(\theta_k,\varphi_k)\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}A_{lm}\Bigg[-iV_{jn}\hat{\mathbf{e}}_j\cdot\mathbf{P}_{lm}(\theta_k,\varphi_k)\mathbf{L}^{(i)}_{lm}(k_n,\mathbf{r}) + \frac{1}{\sqrt{l(l+1)}}[V_{jn}\hat{\mathbf{e}}_j\cdot\mathbf{C}_{lm}(\theta_{k,}\varphi_k)\mathbf{M}^{(j)}_{lm}(k_n,\mathbf{r}) - iV_{jn}\hat{\mathbf{e}}_j\cdot\mathbf{B}_{lm}(\theta_k,\varphi_k)\mathbf{N}^{(i)}_{lm}(k_n,\mathbf{r})]\Bigg] \ , \quad i=1,2,3,4 \ . \tag{19}$$

Following a procedure similar to the one used to derive (16), we have

$$\mathsf{T}^{(i)}_{nl'm'}(\mathbf{r})=\sum_{n=1}^{3}\int_0^{2\pi}d\varphi_k\int_0^{\pi}k_n\sin\theta_k d\theta_k X_{l'm'}(\theta_k,\varphi_k) \times T_{ijn}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}A_{lm}X^{*}_{lm}(\theta_k,\varphi_k)z_l^{(i)}(k_n r)X_{lm}(\theta,\varphi) \quad (i=1,2,3,4) \ . \tag{20}$$

Following a procedure similar to the one used to derive (19), we have the expansion of the traction force vector on a spherical surface in terms of vector spherical wave functions,

$$\mathbf{T}_r(\mathbf{r})=\hat{\mathbf{r}}\cdot\mathsf{T}(\mathbf{r})$$

$$=\sum_{n=1}^{3}\int_0^{\pi}\int_0^{2\pi}X_{l'm'}(\theta_k,\varphi_k)\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}A_{lm}\Bigg[T_{ijn}\hat{\mathbf{e}}_i\cdot\hat{\mathbf{r}}\hat{\mathbf{e}}_j\cdot\mathbf{P}_{lm}(\theta_k,\varphi_k)\mathbf{L}^{(i)}_{lm}(k_n,\mathbf{r}) + \frac{1}{\sqrt{l(l+1)}}[T_{ijn}\hat{\mathbf{e}}_i\cdot\hat{\mathbf{r}}\hat{\mathbf{e}}_j\cdot\mathbf{C}_{lm}(\theta_k,\varphi_k)\mathbf{M}^{(i)}_{lm}(k_n,\mathbf{r}) - iT_{ijn}\hat{\mathbf{e}}_i\cdot\hat{\mathbf{r}}\times\hat{\mathbf{e}}_j\cdot\mathbf{B}_{lm}(\theta_k,\varphi_k)\mathbf{N}^{(i)}_{lm}(k_n,\mathbf{r})]\Bigg] \ , \quad i=1,2,3,4 \ . \tag{21}$$

The components of the tensor T in a spherical system can also be obtained by the coordinate transformation laws given in Auld's book, but the results to be given in the following are simpler.

Note that

$$\hat{\mathbf{r}}=\sin\theta\cos\varphi\hat{\mathbf{x}}+\sin\theta\sin\varphi\hat{\mathbf{y}}+\cos\theta\hat{\mathbf{z}} \ . \tag{22a}$$

Then we have

$$\hat{\mathbf{r}}\cdot\hat{\mathbf{x}}=\sin\theta\cos\varphi=P_1^1(\cos\theta)\cos\varphi \ , \tag{22b}$$

$$\hat{\mathbf{r}}\cdot\hat{\mathbf{y}}=\sin\theta\sin\varphi=P_1^1(\cos\theta)\sin\varphi \ , \tag{22c}$$

$$\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}=\cos\theta=P_1(\cos\theta) \ . \tag{22d}$$

It is also noted that

$$(2l+1)(\sin\theta)e^{j\varphi}X_{lm}(\theta,\varphi)=X_{l+1,m+1}(\theta,\varphi)-X_{l-1,m+1}(\theta,\varphi)\ , \tag{23a}$$

$$(2l+1)(\sin\theta)e^{-i\varphi}X_{lm}(\theta,\varphi)=(l+m)(l+m-1)X_{l-1,m-1}(\theta,\varphi) -(l-m+1)(l-m+2)X_{l+1,m-1}(\theta,\varphi)\ , \tag{23b}$$

$$(2l+1)(\cos\theta)X_{lm}(\theta,\varphi)=(l-m+1)X_{l+1,m}(\theta,\varphi)+(l+m)X_{l-1,m}(\theta,\varphi)\ . \tag{23c}$$

We get

$$\hat{\mathbf{r}}\cdot\hat{\mathbf{x}}\mathbf{P}_{lm}(\theta,\varphi)=\sin\theta\cos\varphi\hat{\mathbf{r}}X_{lm}(\theta,\varphi) =\frac{1}{2(2l+1)}[\mathbf{P}_{l+1,m+1}(\theta,\varphi)-\mathbf{P}_{l-1,m+1}(\theta,\varphi)+(l+m)(l+m-1)\mathbf{P}_{l-1,m-1}(\theta,\varphi) -(l-m+1)(l-m+2)\mathbf{P}_{l+1,m-1}(\theta,\varphi)]\ , \tag{24a}$$

$$\hat{\mathbf{r}}\cdot\hat{\mathbf{y}}\mathbf{P}_{lm}(\theta,\varphi)=\frac{1}{2(2l+1)i}[\mathbf{P}_{l+1,m+1}(\theta,\varphi)-\mathbf{P}_{l-1,m+1}(\theta,\varphi)-(l+m)(l+m-1)\mathbf{P}_{l-1,m-1}(\theta,\varphi) -(l-m+1)(l-m+2)\mathbf{P}_{l+1,m-1}(\theta,\varphi)]\ , \tag{24b}$$

$$\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}\mathbf{P}_{lm}(\theta,\varphi)=\frac{1}{2l+1}[(l-m+1)\mathbf{P}_{l+1,m}(\theta,\varphi)+(l+m)\mathbf{P}_{l-1,m}(\theta,\varphi)]\ . \tag{24c}$$

Similarly, we can expand $\hat{\mathbf{r}}\cdot\hat{\mathbf{x}}\mathbf{B}_{lm}(\mathbf{C}_{lm})$, $\hat{\mathbf{r}}\cdot\hat{\mathbf{y}}\mathbf{B}_{lm}(\mathbf{C}_{lm})$, and $\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}\mathbf{B}_{lm}(\mathbf{C}_{lm})$ in terms of (22) and the following formulas [21,23]:

$$X_p^{\mu-m}(\theta,\varphi)\mathbf{B}_{-\mu v}(\theta,\varphi)=\sum_l \frac{i^{l-v-p}(-1)^{p+\mu}}{2p+1}a(-\mu,v|m,l|p)a(v,l,p)\mathbf{B}_{-ml}(\theta,\varphi) +\sum_l \frac{i^{l-v-p-1}(-1)^{p+\mu}}{2p+1}a(-\mu,v|m,l|p,p-1)b(v,l,p)\mathbf{C}_{-ml}(\theta,\varphi)\ , \tag{25}$$

$$X_p^{\mu-m}(\theta,\varphi)\mathbf{C}_{-\mu v}(\theta,\varphi)=-\sum_l i^{l-v-p-1}\frac{(-1)^{p+\mu}}{2p+1}a(-\mu,v|m,l|p,p-1)b(v,l,p)\mathbf{B}_{-ml}(\theta,\varphi) +\sum_l \frac{i^{l-v-p}(-1)^{p+\mu}}{2p+1}a(-\mu,v|m,l|p)a(v,l,p)\mathbf{C}_{-ml}(\theta,\varphi)\ . \tag{26}$$

The quantities $a(m,n|-\mu,v|p)$, $a(m,n|-\mu,v|p,p-1)$, $a(n,v,p)$, and $b(n,v,p)$ are found in Refs. [21, 23]. Using the above formulas, we can expand $\mathbf{T}_r(\mathbf{r})$ in terms of the complete orthogonal function set on a spherical surface $\{\mathbf{P}_{lm},\mathbf{B}_{lm},\mathbf{C}_{lm}\}$. So we can treat the scattering from an elastic sphere analytically by solving the linear system of equations, each of whose coefficient is a double integral only [16]. However, following the method in the literature [14], the linear system of equations is derived from Galerkin's method, each of whose coefficients is a double integral involving an infinite double series. Our method is more efficient.

From the formalism given in this section, it is easy to see that once the medium parameters are given, the spherical wave functions of all kinds of HEAM are determined.

The addition theorems of spherical wave functions can be derived directly from those for isotropic media [24].

IV. INTEGRAL REPRESENTATIONS OF SPHERICAL WAVE FUNCTIONS

From the derivation of the above section, it is obvious that the integral representations of spherical wave functions of the first kind of HEAM are

$$\varphi_{nl'm'}^{(1)}(\mathbf{r})=\int_0^{\pi}\int_0^{2\pi}X_{l'm'}(\theta_k,\varphi_k)\varphi_n(\theta_k,\varphi_k) \times e^{i\mathbf{k}_n\cdot\hat{\mathbf{r}}}k_n\sin\theta_k d\theta_k d\varphi_k\ . \tag{27}$$

The integral representation of spherical wave functions of the ith kind for HEAM is suggested by the physical insight into the problem [16],

$$\varphi_{nl'm'}^{(i)}(\mathbf{r})=\int_{C_i}\int_0^{2\pi}X_{l'm'}(\theta_k,\varphi_k)\varphi_n(\theta_k,\varphi_k) \times e^{i\mathbf{k}_n\cdot\mathbf{r}}k_n\sin\theta_k d\theta_k d\varphi_k\ , \tag{28}$$

where C_i $(i=1,2,3,4)$ is the complex integral path of spherical wave functions for isotropic media [21,24,25].

In this section, a strictly mathematical proof of (28) will be given. In fact, from (14) we have

$$j_l(k_n r)X_{lm}^*(\theta_k,\varphi_k)=\frac{1}{4\pi i^l}\int_0^{2\pi}\int_0^{\pi}e^{i\mathbf{k}_n\cdot\mathbf{r}}X_{lm}^*(u',v') \times\sin u'du'dv'\ , \tag{29a}$$

$$\mathbf{r}=r[\hat{\mathbf{x}}\sin u'\cos v'+\hat{\mathbf{y}}\sin u'\sin v'+\hat{\mathbf{z}}\cos u']\ . \tag{29b}$$

Equations (14) and (29) can be proved by examining them for every given (θ_k,φ_k). So they hold even if k_n is a function of (θ_k,φ_k).

From (28), using (14) and (29), we find

$$\varphi^{(i)}_{nl'm'}(r,\theta,\varphi)=\int_0^{2\pi}\int_0^{\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}A_{lm}\frac{1}{4\pi i^l}X_{l'm'}(\theta,\varphi)X^*_{lm}(u',v')\varphi^{(i)}_{nl'm'}(r,u',v')\sin u'du'dv',\quad i=1,2,3,4\ . \tag{30}$$

This is the integral equation for the integral representations of the wave functions of the ith kind. Substituting the series expression of the wave functions of the ith kind (16) into Eq. (30), it is an identity. So we prove that Eqs. (16) and (30) are identical. This is the foundation of the derivation of tensor Green's functions in Sec. V of this paper.

Since φ can represent any component of $\mathbf{V}$ and T, we have the integral representations of the ith and tensor spherical wave functions of various kinds for HEAM,

$$\mathbf{V}_{nl'm'}(\mathbf{r})=\int_0^{2\pi}d\varphi_k\int_{C_i}d\theta_k k_n\sin\theta_k V_{jn}\hat{\mathbf{e}}_j e^{i\mathbf{k}_n\cdot\mathbf{r}}\ ,\qquad i=1,2,3,4 \tag{31a}$$

$$\mathsf{T}_{nl'm'}(\mathbf{r})=\int_0^{2\pi}d\varphi_k\int_{C_i}d\theta_k k_n\sin\theta_k T_{ijn}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j e^{i\mathbf{k}_n\cdot\mathbf{r}}\ ,\qquad i=1,2,3,4\ , \tag{31b}$$

where C_i is the complex integral path of spherical wave functions for isotropic media [24,25].

V. DYADIC GREEN'S FUNCTIONS IN HEAM

Using the identity

$$\delta(\mathbf{r}-\mathbf{r}')=\frac{1}{8\pi^3}\int_{-\infty}^{+\infty}d^3k e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\ , \tag{32a}$$

the Fourier transformation of the tensor Green's function reads [4]

$$\mathsf{G}(\mathbf{k},\mathbf{r}')=\int_{-\infty}^{+\infty}d^3r\mathsf{G}(\mathbf{r},\mathbf{r}')e^{-i\mathbf{k}\cdot\mathbf{r}}=\mathsf{C}^{-1}e^{-i\mathbf{k}\cdot\mathbf{r}'}\ , \tag{32b}$$

where [see (7)]

$$\mathsf{C}=k^2\begin{bmatrix}\alpha & \delta & \epsilon\\ \delta & \beta & \zeta\\ \epsilon & \zeta & \gamma\end{bmatrix}-\rho\omega^2\mathsf{I}\ . \tag{32c}$$

The dispersion relation can be considered as a general eigenvalue problem of real symmetrical matrices, so we have [26]

$$\mathsf{G}(\mathbf{k},\mathbf{r}')=\sum_{n=1}^{3}\frac{\mathbf{V}_n\mathbf{V}_n e^{-i\mathbf{k}\cdot\mathbf{r}'}}{(k^2-k_n^2)N_n^2}\ , \tag{33}$$

where k_n and $\mathbf{V}_n$ are the characteristic wave number and wave vector of the nth characteristic wave, respectively, as discussed in Sec. II, and N_n is the normalized value of the nth eigenwave vector [26].

The inverse Fourier transform of Eq. (33) is

$$\mathsf{G}(\mathbf{r},\mathbf{r}')=\int_{-\infty}^{+\infty}d^3k\sum_{n=1}^{3}\frac{\mathbf{V}_n\mathbf{V}_n e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{(k^2-k_n^2)N_n^2}\ . \tag{34}$$

The general representations in a spherical coordinate system to be derived in this section have not been obtained so far, although many researchers have treated the problem in a rectangular coordinate system [4,5].

From

$$\mathbf{V}(\mathbf{r})=\int\mathsf{G}(\mathbf{r},\mathbf{r}')\cdot\mathbf{f}(\mathbf{r}')d\mathbf{r}' \tag{35}$$

we know that $\mathsf{G}(\mathbf{r},\mathbf{r}')$ represents outgoing waves which can be expanded by the spherical wave functions of the third kind for HEAM, in which the integral for θ_k is along the same path C_3 as shown in Sec. IV. Therefore, in (34) the integral over θ_k is along the path C_3. This conclusion is supported by Weyl in calculating the point-source radiation in isotropic media [25]. Since $k_\rho=k_n\sin\theta_k$ and k_n is a bounded function of θ_k and φ_k if and only if the integration path C_3 for θ_k is chosen, the correct complex integration path for k_ρ may be obtained as shown by Weyl [25]. So we have

$$\begin{aligned}\mathsf{G}(\mathbf{r},\mathbf{r}')&=\frac{1}{8\pi^3}\int_{-\infty}^{+\infty}d^3k\mathsf{G}(k,\mathbf{r}')\cdot e^{i\mathbf{k}\cdot\mathbf{r}}\\&=\frac{1}{8\pi^3}\int_0^{2\pi}d\varphi_k\int_{C_3}d\theta_k\int_{-\infty}^{+\infty}\sum_{n=1}^{3}\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\mathbf{V}_n\mathbf{V}_n}{(k^2-k_n^2)N_n^2}k^2dk\sin\theta_k\\&=\frac{i}{8\pi^2}\int_0^{2\pi}\int_{C_3}d\varphi_k d\theta_k\sum_{n=1}^{3}\frac{e^{ik_n\cdot(\mathbf{r}-\mathbf{r}')}}{N_n^2}\mathbf{V}_n\mathbf{V}_n(\sin\theta_k)k_n\ .\end{aligned} \tag{36}$$

In Secs. III and IV, we show two relations. The first relation is the one between integral representation and series representation of spherical wave functions of the third kind for HEAM. The second relation is the one between the spherical wave functions for HEAM of the third kind and that of the first kind. In view of the above relations, we obtain

$$\mathsf{G}(\mathbf{r},\mathbf{r}')=\frac{i}{8\pi^2}\sum_{n=1}^{3}\int_0^{2\pi}d\varphi_k\int_0^{\pi}d\theta_k\sum_{l=1}^{\infty}\sum_{m=-l}^{+l}A_{lm}A^*_{lm}(\theta_k,\varphi_k)\frac{\mathbf{V}_n\mathbf{V}_n}{N_n^2}h_l^{(1)}(k_n|\mathbf{r}-\mathbf{r}'|)P_l^m(\cos\theta')e^{im\phi'}\sin\theta_k k_n \tag{37}$$

where (θ',φ') means $\theta_{rr'}$, $\varphi_{rr'}$ [21], and A_{lm} are defined by Eq. (14). Practically, $V(\mathbf{r})$ is computed by substituting (37) into (35).

Equation (37) can be written in the form of separation of variables [24] when $r<r'$ and $r>r'$. The Green's tensor [6]

$$\sum_{i,j,m}=C_{ijkl}(\partial_k G_{lm}) \tag{38}$$

can be easily derived by using (34) and (38). The expression for $\sum_{ijm}$ is of a similar form as (37), which is useful in the boundary integral equation and T-matrix formulations [6].

VI. SUMMARY AND DISCUSSION

We have developed an alternative method to transform the eigenwave theory of unbounded HEAM to the eigenfunction theory of bounded HEAM. We have also found a unified method to study the dyadic Green's functions in both isotropic and anisotropic elastic media. This study shows that for an arbitrary homogeneous elastic region, the acoustic fields can be expanded by a series of the eigenwave functions, each of which is also a series just like the isotropic case in the spheroidal coordinate system [24]. So we can use the simple method of mode matching to analyze the guidance, resonance, radiation, and scattering in HEAM. The theory developed in this paper is applicable to bounded coordinate systems, such as elliptic cylindrical and spheroidal coordinate systems. The canonical solutions of wave functions and dyadic Green's functions for HEAM given in this paper are useful in the further study of the problems in elastic spherical layered structures.

As the eigenwave theory of unbounded elastic media and the far-region fields of the Green's functions are the tools for the analysis of the linear and weakly nonlinear problems, respectively [4], the wave-function theory of bounded HEAM and the general representations of Green's functions given in this paper are the foundation of the corresponding problems [4,9]. The latter is also useful in the boundary integral equations and T-matrix formulation of the problems [6].

The present work, in comparison with previous works, has the following features.

(i) Although the method of angular spectrum expansion has been successfully used to solve the problems in a simply connected domain by many authors [10–15], we generalize the treatment of the electromagnetic fields in the anisotropic annular domain [16] to that of the acoustic fields in elastic anisotropic annular domain. While all the authors of Refs. [10–15] used the representations of the plane-wave angular spectrum to simplify the numerical computation, we utilize the similar representations for the general solutions of the anisotropic elastodynamic wave equations in HEAM. The focus point is therefore different. Furthermore, there are similarities between the present theory and the classical one [20,25] for isotropic media. This is helpful in the formulation of boundary value problems. Moreover, invoking the asymptotic expression of Bessel functions for large order [27], we can easily treat the truncation of series appearing in this paper.

(ii) From the view of mathematical physics, our theory overcomes the difficulty of solving the anisotropic elastodynamic wave equations in spherical coordinate systems by separation of variables, but the essence of the present theory is the physical insight into the physical problem. This point was further shown in deriving the Green's functions by means of Weyl's method. It is also interesting that the elastodynamic wave equations for HEAM are exactly solved without resorting to the coupled differential equations in a spherical coordinate system directly. This is something like Hansen's solutions to the vector wave equations in isotropic media [25]. However, Hansen's solutions are also based on the differential equations directly. Thus the solution given in this paper is in a different light.

(iii) The expansion of the undetermined angular spectrum amplitudes for isotropic media is well known [20,25], and it is well motivated to pursue such an approach, but to the best of our knowledge this paper represents the first attempt to do so for the HEAM [16] whose eigenwave numbers are functions of the directions of the wave vector. The successive steps are not very novel, but it was considered worthwhile to examine the elastodynamic waves in HEAM and to spell them out for reference purposes.

(iv) The formulation of the present theory facilitates the utilization of the character of media [16]. For example, for the homogeneous elastic transversely isotropic media [8,28], the eigenwave numbers are independent of φ_k and all the integrals about φ_k appearing in this paper can be evaluated analytically in terms of Eq. (23).

There are many topics in the elastic wave theory of bounded homogeneous anisotropic media to be studied. Some of them will be left for future papers on applications.

APPENDIX: VECTOR SPHERICAL WAVE FUNCTIONS

The spherical vector wave functions are [20]

$$\begin{aligned}\mathbf{L}_{lm}^{(i)}(k_n r,\theta,\varphi)&=\frac{1}{k_n}\nabla\Psi_{lm}^{(i)}\\&=\mathbf{P}_{lm}(\theta,\varphi)\frac{1}{k_n}\frac{d}{dr}[z_l^{(i)}(k_n r)]\\&\quad+\sqrt{l(l+1)}\mathbf{B}_{lm}\frac{1}{k_n r}[z_l^{(i)}(k_n r)]\ ,\end{aligned} \tag{A1a}$$

$$\mathbf{M}_{lm}^{(i)}(k_n r,\theta,\varphi)=\nabla\times(\mathbf{r}\Psi_{lm})=\sqrt{l(l+1)}\mathbf{C}_{lm}z_l^{(i)}(k_n r)\ , \tag{A1b}$$

$$\begin{aligned}\mathbf{N}_{lm}^{(i)}(k_n r,\theta,\varphi)&=\frac{1}{k_n}\nabla\times\mathbf{M}_{lm}^{(i)}\\&=\sqrt{l(l+1)}\mathbf{P}_{lm}\frac{1}{k_n r}[z_l^{(i)}(k_n r)]\\&\quad+\sqrt{l(l+1)}\mathbf{B}_{lm}\frac{1}{k_n r}\frac{d}{dr}[z_l^{(i)}(k_n r)]\ ,\end{aligned} \tag{A1c}$$

where

$$\Psi_{lm}^{(i)}=z_l^{(i)}(k_n r)P_l^m(\cos\theta)e^{im\varphi} , \tag{A2a}$$

$$z_l^{(i)}(k_n r)=\begin{cases} j_n(k_n r), & i=1 \\ y_n(k_n r), & i=2 \\ j_n(k_n r)+iy_n(k_n r), & i=3 \\ j_n(k_n r)-iy_n(k_n r), & i=4 \end{cases} \tag{A2b}$$

and

$$\mathbf{P}_{lm}(\theta,\varphi)=\hat{\mathbf{r}}X_{lm}(\theta,\varphi)=\hat{\mathbf{r}}P_l^m(\cos\theta)e^{im\varphi} , \tag{A3}$$

$$\mathbf{B}_{lm}(\theta,\varphi)=r\nabla X_{lm}(\theta,\varphi)=\mathbf{r}\times\mathbf{C}_{lm}(\theta,\varphi) =\hat{\boldsymbol{\theta}}\frac{dP_l^m(\cos\theta)}{d\theta}e^{im\varphi}+\hat{\boldsymbol{\phi}}\frac{im}{\sin\theta}P_l^m(\cos\theta)e^{im\varphi} , \tag{A4}$$

$$\mathbf{C}_{lm}(\theta,\varphi)=\hat{\boldsymbol{\theta}}\frac{im}{\sin\theta}P_l^m(\cos\theta)e^{im\varphi}-\hat{\boldsymbol{\varphi}}\frac{dP_l^m(\cos\theta)}{d\theta}e^{im\varphi} . \tag{A5}$$

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